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SURFACE TEMPERATURE CALCULATIONS IN RADIANT SURROUNDINGS OF ARBITRARY COMPLEXITY—FOR GRAY, DIFFUSE RADIATION

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Abstract—The question of calculating the temperatures of opaque and non-opaque surfaces subject to assigned net rates of radiant flux is considered. It is shown that, for gray surfaces, any opaque surface in radiant balance in an enclosure of arbitrary complexity achieves a steady state temperature which is independent of the emissivity (or absorptivity) of the surface. Relations are developed for the temperature of opaque and non-opaque surfaces subjected to an assigned net radiant flux.

Résumé—On considère le calcul des températures de surfaces, opaques ou non, soumises à des densités de flux de rayonnement bien déterminées. On montre que, dans le cas de corps gris, une surface opaque quelconque, en équilibre de rayonnement dans une enceinte de complexité arbitraire, acquiert, en régime permanent, une température qui est indépendante de ses coefficients d'absorption ou d'émission. Des relations sont données pour la température de surfaces, opaques ou non, soumises à des flux de rayonnement bien déterminés.

Zusammenfassung—Die Berechnung von Temperaturen strahlungsundurchlässiger und durchlässiger Oberflächen bei gegebener Strahlungsmenge wird untersucht. Jede undurchlässige Oberfläche eines grauen Strahlers in einer willkürlich aufgebauten Umhüllung nimmt im Strahlungsgleichgewicht eine gleichmässige Endtemperatur an, unabhängig vom Emissionsvermögen (oder Absorptionsvermögen) der Oberfläche. Die Abhängigkeit der Temperatur undurchlässiger und durchlässiger Oberflächen von der zugeführten Strahlungsmenge ist angegeben.

Аннотация—Рассматривается проблема вычисления температур непрозрачных и прозрачных поверхностей, через которые проходит определённый результирующий поток излучения. Показано, что для серых поверхностей любая непрозрачная поверхность в условиях равновесия при окружающей системе произвольной сложности достигает стационарной температуры, которая не зависит от излучательной (или поглощательной) способности поверхности. Выведены соотношения для нахождения температуры непрозрачных и прозрачных поверхностей при прохождении определённого результирующего потока излучения.

NOTATION

- a, absorptivity (absorptance) of a surface;
- A, radiating area of a surface;
- B_{ij}, absorption factor;
- F_{pi}, angle factor, the fraction of the emission of surface p directly incident upon surface i;
- q_j, net rate of radiant energy loss from surface j;
- T, absolute temperature;
- α_{pi}, $= F_{pi}\rho_i$;
- ϵ, total hemispherical emissivity (emittance) of a surface;
- ρ, reflectivity (reflectance) of a surface;
- σ, universal constant for thermal radiation.

INTRODUCTION

IN ENGINEERING practice there are many circumstances in which thermal radiant energy exchange rates may be calculated with sufficient accuracy under the assumptions of gray, diffuse radiation *in vacuo*. The absorption factor method developed and elaborated upon by the present writer [1–3] is well adapted to such calculations and applies to enclosures of arbitrary complexity. Although this method requires the solution of a number of simultaneous linear equations (as must any method which accounts for multiple inter-reflections) the method is conceptually simple and is well suited to machine techniques of equation solution or matrix inversion.

The absorption factor method also permits, due to its simplicity, the demonstration of many of the important characteristics of radiant exchange processes. In the original paper, for example, absorption factor reciprocity was shown to be a simple consequence of the assumptions of gray, diffuse radiation and of uniform irradiation without reference to the second law of thermodynamics.

The present paper is concerned with the application of this method to calculations concerning opaque and non-opaque surfaces in radiant surroundings of arbitrary complexity which are subject to assigned rates of net radiant flux. Such a surface condition appears in many cases of modern technology; for example, in electrically heated walls and windows, in reactor elements in the absence of a coolant, in electrical circuit elements in a vacuum, and for surfaces subject to solar or other irradiation.

An interesting particular case, the opaque surface in radiant balance, sometimes called an "adiabatic" surface, is treated in detail. This condition often appears in equipment such as: gas turbine and jet engine combustors, furnaces, electronic components, electronic equipment, and satellite and spacecraft components. It is proven that the temperature of a surface in radiant balance is independent of its surface properties. This characteristic is sometimes alluded to, e.g. Eckert [4] in the language of Oppenheim's method [5] of analysis, but has not been proven.

The assumed condition of gray, diffuse radiation and reflection is met in many circumstances of practical importance. Many studies have indicated that even highly non-gray, specular surfaces are often essentially gray and diffuse under the conditions of actual use as a result of surface coatings, corrosion, erosion, or other types of surface alteration.

GENERAL RELATIONS

Any surface A_j in an "enclosure"* of n gray surfaces loses radiant energy at the net rate of q_j,

$$\begin{aligned} q_j &= W_jA_j - B_{1j}W_1A_1 - B_{2j}W_2A_2 - \dots \\ &= W_jA_j - \sum_{=1}^{n} B_{ij}W_iA_i \end{aligned} \tag{1}$$

where B_{ij}, the absorption factor, is the fraction of the radiant energy arising at surface A_i which is absorbed at surface A_j. WA denotes the rate at which energy is radiated from a surface to the space above it (other than by reflection). For an opaque surface, WA becomes merely the hemispherical emissive power $\epsilon\sigma T^4A$ where ϵ is the total hemispherical emissivity.

For diffuse radiation and reflection, the absorption factors are given by:

$$\left.\begin{array}{l} (a_{11}-1)B_{1j} + a_{12}B_{2j} + \dots + a_{1n}B_{nj} + F_{1j}\epsilon_j = 0 \\ a_{21}B_{1j} + (a_{22}-1)B_{2j} + \dots + a_{2n}B_{nj} + F_{2j}\epsilon_j = 0 \\ \cdot \\ \cdot \\ \cdot \\ a_{n1}B_{1j} + a_{n2}B_{2j} + \dots + (a_{nn}-1)B_{nj} + F_{nj}\epsilon_j = 0 \end{array}\right\} \tag{2}$$

$$a_{pi} = F_{pi}\rho_i$$

where F_{pi} is the fraction of the emission of surface A_p directly incident upon A_i, ϵ_i is the total hemispherical emissivity of surface A_i, and ρ_i is the reflectivity of surface A_i. In general,

* The term "enclosure" is meant to denote the assembly of all radiant conditions relevant to the surface under consideration. Under this definition of an enclosure, all radiation circumstances are "enclosure" problems.

opaque surfaces in radiant balance, other than A_j, are assigned a reflectivity of 1·0.*

The absorption factor relations may be written more compactly as:

$$\sum_{i=1}^{n} (a_{pi} - \delta_{pi})B_{ij} + F_{pj}\epsilon_j = 0 \tag{3}$$

where δ_{pi} is Kronecker's delta. The reciprocity relations among the n^2 values of B_{ij} for any combination of opaque and non-opaque surfaces are:

$$\epsilon_i B_{ij} A_i = \epsilon_j B_{ji} A_j. \tag{4}$$

* Since the temperature of such a surface may not be arbitrarily specified, the total hemispherical emissive power is not known. This assumption, of unit reflectivity, satisfies the condition of radiant balance and removes the unknown from the equation, e.g. equation (1).

TEMPERATURE OF AN OPAQUE SURFACE IN RADIANT BALANCE

Taking A_j as an opaque surface in radiant balance, denoted hereafter as A_a, the value of q_a is zero. By equations (1) and (4), we have

$$\begin{aligned} q_a &= W_a A_a - \sum_{i=1}^{n} B_{ia} W_i A_i \\ &= A_a \epsilon_a \sigma(T_a^4 - \sum_{i=1}^{n} B_{ai} T_i^4) = 0. \end{aligned} \tag{5}$$

The B_{ai} are given by (2) or (3) as

$$B_{ai} = \frac{D_{ai}}{D} \tag{6}$$

where D_{ai} and D denote the following two determinants:

$$D_{ai} = \begin{vmatrix} (a_{11}-1) & a_{12} & \cdots & a_{1,(a-1)} & -F_{1i}\epsilon_i & a_{1,(a+1)} & \cdots & a_{1n} \\ a_{21} & (a_{22}-1) & \cdots & a_{2,(a-1)} & -F_{2i}\epsilon_i & a_{2,(a+1)} & \cdots & a_{2n} \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ a_{n1} & & & & & & & (a_{nn}-1) \end{vmatrix} \tag{7}$$

$$D = \begin{vmatrix} (a_{11}-1) & a_{12} & \cdots & a_{1,(a-1)} & a_{1a} & a_{1,(a+1)} & \cdots & a_{1n} \\ a_{21} & (a_{22}-1) & \cdots & a_{2,(a-1)} & a_{2a} & a_{2,(a+1)} & \cdots & a_{2n} \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ a_{n1} & & & & & & & (a_{nn}-1) \end{vmatrix}. \tag{8}$$

Combining (5) and (6), we obtain

$$T_a^4 = \frac{\sum_i D_{ai} T_i^4}{(D - D_{aa})}; \quad i = 1, 2, \ldots, (a-1), (a+1), \ldots, n \tag{9}$$

It is seen from equation (7) that ϵ_a, which is equal to $(1 - \rho_a)$, does not appear in the D_{ai} for $i \neq a$. Therefore, the numerator of equation (9) does not depend upon ϵ_a. D and D_{aa} are written as follows, moving the ath column into the first column position in both determinants.

$$D = (-1)^{a-1} \begin{vmatrix} F_{1a}\rho_a & (a_{11}-1) & a_{12} & \dots & a_{1n} \\ F_{2a}\rho_a & a_{21} & (a_{22}-1) & \dots & a_{2n} \\ \vdots & & & & \\ (F_{aa}\rho_a - 1) & a_{a1} & a_{a2} & \dots & a_{an} \\ \vdots & & & & \\ F_{na}\rho_a & a_{n1} & a_{n2} & \dots & (a_{nn}-1) \end{vmatrix}$$

$$D_{aa} = (-1)^{a-1} \begin{vmatrix} -F_{1a}\epsilon_a & (a_{11}-1) & a_{12} & \dots & a_{1n} \\ -F_{2a}\epsilon_a & a_{21} & (a_{22}-1) & \dots & a_{2n} \\ \vdots & & & & \\ -F_{na}\epsilon_a & a_{n1} & a_{n2} & \dots & (a_{nn}-1) \end{vmatrix}.$$

Both determinants are the same except for the first column. Call N_k the value of the minor of the term in the first column in the kth row. Note that none of the N_k contain ϵ_a or ρ_a. The kth term of $(D - D_{aa})$ where $k \neq a$ is written as

$$(-1)^{k+1}(-1)^{a-1}(F_{ka}\rho_a N_k + F_{ka}\epsilon_a N_k) = (-1)^{k+1}(-1)^{a-1}N_k F_{ka}. \tag{10}$$

None of these terms contain ϵ_a. For the term $k = a$,

$$(-1)^{a+1}(-1)^{a-1}[(F_{aa}\rho_a - 1)N_a + F_{aa}\epsilon_a N_a] = (-1)^{a+1}(-1)^{a-1}N_a(F_{aa} - 1). \tag{11}$$

Again ϵ_a is absent. Therefore, $(D - D_{aa})$ is independent of ϵ_a. Since neither the numerator nor the denominator depend upon ϵ_a (or ρ_a), T_a is independent of the surface properties of A_a and depends only upon the temperature and emissivities or surface conditions of the other surfaces of the radiant surroundings.

The results in (10) and (11) indicate that the denominator of equation (9) may be written in the following simpler form for calculations;

$$D_a = D - D_{aa} = (-1)^{a-1} \begin{vmatrix} F_{1a} & (a_{11}-1) & a_{12} & \dots & a_{1n} \\ F_{2a} & a_{21} & (a_{22}-1) & \dots & a_{2n} \\ \vdots & & & & \\ (F_{aa}-1) & a_{a1} & a_{a2} & \dots & a_{an} \\ \vdots & & & & \\ F_{na} & a_{n1} & a_{n2} & \dots & (a_{nn}-1) \end{vmatrix}. \tag{12}$$

The foregoing analysis shows that the temperature of an opaque surface in radiant balance is independent of its surface property emissivity, i.e. the temperature may be calculated from equation (9) using any value of ϵ_a in equations (7) and (8). However, it is not necessary to assume a value of ϵ_a since equations (12) and (9) may be used. This same result may be expressed as follows:

$$T_a^4 = \frac{\sum_i^n D_{ai}T_i^4}{(-1)^{a-1}\left[(-1)^{a+1}(F_{aa}-1)N_a + \sum_k^n (-1)^{k+1} F_{ka}N_k\right]};$$

$$i \neq a \text{ and } k \neq a. \quad (13)$$

OPAQUE SURFACES WITH ASSIGNED NET RADIANT LOSS

A surface (in a radiant surrounding of arbitrary complexity) whose temperature level adjusts to result in a given net rate of energy loss by radiation amounts to the assignment of a given value of q_a. The necessary temperature level is found from equation (5) as

$$T_a^4 = \frac{\sum_i^n D_{ai}T_i^4}{D_a} + \frac{q_a D}{A_a\epsilon_a\sigma D_a}; \quad i \neq a. \quad (14)$$

Although the first term on the right is independent of ϵ_a, the second is not and the resulting temperature depends upon ϵ_a. Equation (14) is suitable for calculations.

NON-OPAQUE SURFACES

The general formulations, equations (1), (2) and (3), apply for enclosures which contain windows, openings, or any other surfaces whose radiation characteristics are gray and diffuse. For any surface A_j the equations take into account the effects of the radiant energy transmission across all non-opaque surfaces under the assumption that none of the thermal radiation transmitted outside of the enclosure is again incident upon the outside of the transmitting areas due, for example, to reflections. This is a reasonable assumption in many practical circumstances.

If A_j is opaque, then q_j is the net rate of radiant energy loss from A_j. If A_j is non-opaque, q_j is the net rate of radiant energy loss from the "enclosure" side of A_j. The rate of energy transmission through A_j is

$$(q_T)_j = \frac{\tau_j}{a_j}\sum_{i=1}^{n} B_{ij}W_iA_i = \tau_j\sigma A_j \sum_{i=1}^{n} B_{ji}T_i^4 \quad (15)$$

where τ_j and a_j are, respectively, the transmissivity and absorptivity of A_j.

The calculation of the temperature of a non-opaque surface with an assigned net (or zero) radiant energy loss rate takes into account both sides of the surface and the two different enclosures A and B with which the two sides exchange energy. The n surfaces in enclosure A are numbered 1, 2, . . . , i, . . . , n and the N surfaces in B are numbered 1, 2, . . . , m, . . . , N. The surface under consideration is denoted by a and its net rate of radiant energy loss is

$$q_a = (q_a)_A + (q_a)_B = (W_aA_a - \sum_{i=1}^{n} B_{ia}W_iA_i)_A + (W_aA_a - \sum_{m=1}^{N} B_{ma}W_mA_m)_B. \quad (16)$$

If the two sides of surface a are of equal area and emissive power, this result may be written as

$$q_a = 2W_aA_a - \sum_{i=1}^{n} B_{ia}W_iA_i - \sum_{m=1}^{N} B_{ma}W_mA_m. \quad (17)$$

Employing the reciprocity relation and solving for T_a^4, we have

$$[2 - (B_{aa})_A - (B_{aa})_B]\, T_a^4 = \frac{q_a}{\epsilon_a\sigma A_a} + \sum_i^n B_{ai}T_a^4 + \sum_m^N B_{am}T_m^4 \quad (18)$$

where $i \neq a$ and $m \neq a$.

For the case of a non-opaque surface in radiant balance, q_a is zero. However, it is not true that T_a is independent of ϵ_a for a non-opaque surface.

CONCLUSION

The foregoing analysis has shown that the temperature achieved by an opaque surface in radiant balance is independent of its surface properties. Relations are derived for temperature calculations for opaque and non-opaque surfaces subject to an assigned net thermal radiant exchange rate.

Although the absorption factor method may be used in analysing enclosures containing emitting and absorbing media, the present treatment applies only in the absence of such radiation effects. However, if the principal effect of such an intervening medium is convection at the various enclosure surfaces, its presence may be simply accounted for by an iterative method of calculating the temperature of an assigned net flux surface. A temperature is assumed, the rate of convection loss is calculated and subtracted from the assigned rate of loss to obtain the net rate of radiant loss. From this, a surface temperature is computed which is compared with the assumed value.

This absorption factor method applies, within the limits of the initial assumptions, to enclosures of arbitrary complexity, i.e. for arbitrarily large values of n. It has been pointed out [3] that the inaccuracies inherent in the assumption of uniform irradiation of each surface by each other surface may be reduced continually toward zero by further subdivision of the surfaces of the enclosure.*

If a given area in radiant balance in an enclosure is subject to highly non-uniform irradiation, its temperature will be far from uniform. More accurate estimates of the temperature achieved by such a surface may be obtained by subdividing it into separate zones for which individual temperatures are calculated. By this means a temperature distribution is obtained.

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* This subdivision procedure must be judiciously employed, however. The number of numerical operations which must be carried out to obtain a solution is proportional to n^3.